INTERACTION OF A SYSTEM OF GRIFFITH CRACKS IN AN ELASTO-BRITTLE MATERIAL

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Problems of the theory of cracks have recently attracted a great deal of attention. Finding accurate solutions for systems of cracks presents considerable mathematical difficulties. An approximate solution of a static problem of the theory of equilibrium cracks is given in [1]. The hypothesis formulated in [1] is used in this article in which the problem of interaction of Griffith cracks in an elasto-brittle material is analyzed and the accuracy of the solution obtained is estimated.

Let an infinite isotropic body contain an infinitely large number of cracks of a length $2l$, parallel to the axis of abscissas and spaced at a distance of 2h from each other. A constant pressure p is acting inside each crack along the length $2l$. In view of the symmetry of the system, our considerations may be confined to a band $0 \leq y \leq h$ whose lower edge coincides with the longitudinal crack axis, its upper edge being halfway between two adjacent cracks. The problem is to find the relationship between p, h and l if all the elastic constants and the cohesion modulus are known.

Let us consider plane deformation. In this case the stress tensor components σ_x, σ_y, and σ_{xy} and displacement vector components u and v are expressed through two analytical functions $\varphi(z)$ and $\psi(z)$ and their derivatives by the Kolosov-Kuskhelishvili formulas:

$$\sigma_x + \sigma_y = 2\,[\varphi'(z) + \overline{\varphi'(z)}], \qquad (1)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2\,[\bar{z}\varphi''(z) + \psi'(z)], \qquad (2)$$

$$2\mu\,(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \qquad (\varkappa = 3 - 4\nu). \qquad (3)$$

Here ν is the Poisson ratio and μ shear modulus.

The boundary conditions will be

$$\sigma_y = -p \quad (y = 0, \quad -l \leqslant x \leqslant +l), \qquad (4)$$

$$\sigma_{xy} = 0 \qquad (y = 0,\ y = h, -\infty < x < +\infty), \qquad (5)$$

$$v = 0 \quad \begin{cases} (y = 0, & -\infty < x < -l,\ +l < x < +\infty) \\ (y = h, & -\infty < x < +\infty). \end{cases} \qquad (6)$$

The stress tensor components at infinity behave as

$$\sigma_{ij}(r) = o\,(r^{-1}) \qquad \text{at} \quad r \to \infty. \qquad (7)$$

Introducing in accordance with [1] an assumption

$$\sigma_x = \sigma_y \qquad \text{at} \quad y = 0\,, \qquad (8)$$

we can formulate the problem in terms of one analytical function $\varphi'(z)$:

$$\begin{aligned} &\operatorname{Re}\varphi'(z) = -{}^1/_2 p \quad (y = 0,\ -l \leqslant x \leqslant +l) \\ &\operatorname{Im}\varphi'(z) = 0 \quad (y = 0 - \infty < x < -l, \\ &\qquad\qquad\qquad +l < x < +\infty) \\ &\operatorname{Im}\varphi'(z) = 0 \quad (y = h,\ -\infty < x < +\infty)\,. \end{aligned} \qquad (9)$$

A conformal mapping of the band $0 \leq y \leq h$ on the upper semiplane (Fig. 2) is given by

$$\zeta = e^{\pi z/h}\,. \qquad (10)$$

In the semi-plane $\eta > 0$ we obtain the complex Keldysh-Sedov boundary problem [3].

The unknown analytical function $\varphi'(z)$ should have singularities of the order of $x^{-1/2}$ at points $z = l$ and $z = +l$ approached from the left and right, respectively, and the coefficients at these singularities should—in view of the symmetry of the problem—be equal.

A solution satisfying these conditions is in the form

$$\varphi'(z) = -\frac{p}{2}\left\{1 - \frac{\sqrt{a}}{a+1}\left[\sqrt{a}\left(\frac{e^{\pi z/h} - a^{-1}}{e^{\pi z/h} - a}\right)^{1/2} + \frac{1}{\sqrt{a}}\left(\frac{e^{\pi z/h} - a}{e^{\pi z/h} - a^{-1}}\right)^{1/2}\right]\right\}. \quad (a = e^{\pi l/h}) \qquad (11)$$

Using Eqs. (8) and (1), we find that at the crack tips σ_y has singularities in the form

$$\sigma_y(-l - x) = \sigma_y(l + x) = \frac{N}{\sqrt{x}} - \frac{p}{2} + O\,(x^{1/2}),$$

$$N = p\left(\frac{h}{\pi}\right)^{1/2}\left(\frac{e^{\pi l/h} - 1}{e^{\pi l/h} + 1}\right)^{1/2}. \qquad (12)$$

In accordance with [4] we have an equation

$$p\left(\frac{h}{\pi}\right)^{1/2}\left(\frac{e^{\pi l/h} - 1}{e^{\pi l/h} + 1}\right)^{1/2} = \frac{K}{\pi}\,, \qquad (13)$$

where K is the cohesion modulus.

Equation (13) determines the values of p corresponding to the limiting equilibrium of the system of cracks under consideration. After some elementary transformations we obtain an expression relating the half-length of an equilibirum crack to p and h

$$l = \frac{h}{\pi}\ln\frac{p^2 + K^2/\pi h}{p^2 - K^2/\pi h}\,. \qquad (14)$$

At the limit, i.e., at $h \to \infty$, we obtain an expression cited in [4],

$$l = 2K^2/\pi^2 p^2 \qquad (15)$$

relating to an isolated Griffith crack.

It follows from Eq. (14) that for any finite h the pressure p corresponding to infinitely large crack lengths l approaches its critical value

$$p_* = \frac{1}{\sqrt{\pi h}}\,K\,,$$

which increases with decreasing h, while from Eq. (15) it follows that at large l the pressure $p \to 0$.

It follows from Fig. 3 that a system of Griffith cracks spaced at 2h intervals can exist only when $p > p_*$. At $p < p_*$ reversible cracks close up. At equal values of p acting inside cracks, the length of an isolated crack will be smaller than that of a crack in a system of cracks. The pressure p which at a finite h corresponds to the limiting equilibrium is larger for a system of cracks than for an isolated crack (of the same length), this being the way in which the interaction of a system of cracks is manifested.

From Eqs. (1) and (2) we obtain

$$\sigma_y = 2\operatorname{Re}\varphi'(z) + \operatorname{Re}\,[\bar{z}\varphi''(z) + \psi'(z)],$$

$$\sigma_x = 2\operatorname{Re}\varphi'(z) - \operatorname{Re}\,[\bar{z}\varphi''(z) + \psi'(z)]. \qquad (16)$$

Taking into account that $\sigma_{xy} = 0$ at $y = 0$, we find that the hypothesis [8] is equivalent to an equation

$$\bar{z}\varphi''(z) + \psi'(z) = 0 \qquad \text{at} \quad y = 0$$

which is rigorously satisfied only in the case of a single crack.

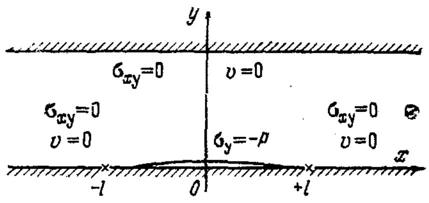

Fig. 1

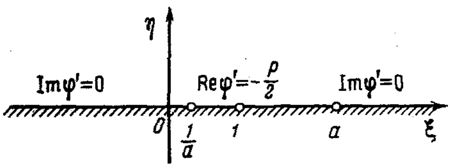

Fig. 2

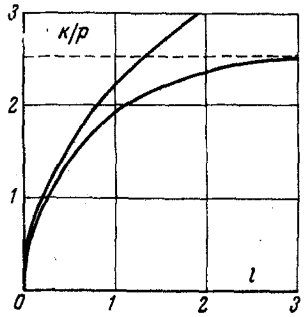

Fig. 3

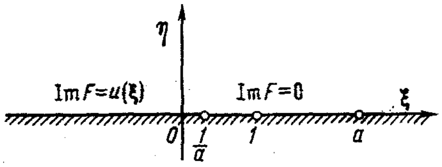

Fig. 4

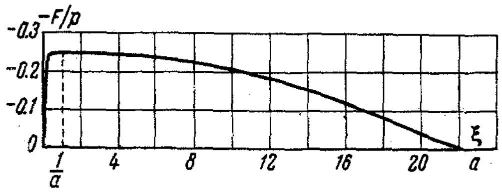

Fig. 5

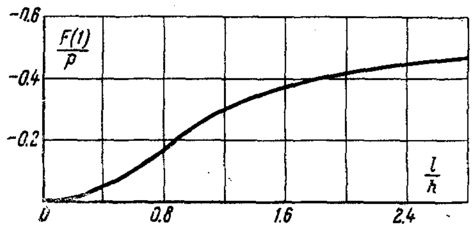

Fig. 6

In the case of a system of cracks $\mathrm{Re}\,[\bar{z}\varphi''(z)+\psi'(z)]$ characterizes the redistribution of stress due to the presence of adjacent cracks. Let us find the values of the ratio l/h at which the above obtained approximate solution may be used.

To this end let us formulate a problem for which an accurate solution can be found.

Consider an infinitely elastic body containing a system of parallel Griffith cracks. Stresses acting in the body are such that along each crack $\sigma_x + \sigma_y = -2p$. The crack length and the spacing between cracks are $2l$ and $2h$, respectively. In view of the symmetry of the problem let us consider only the band $0 \le y \le h$.

The boundary conditions will be

$$\sigma_x+\sigma_y=-2p \qquad (y=0,\ -l\leqslant x\leqslant +l)$$
$$\sigma_{xy}=0 \qquad (y=0,\ y=h,\ -\infty<x<+\infty)$$
$$v=0\begin{cases}(y=0,\ -\infty<x<-l,\ +l<x<+\infty)\\ (y=h,\ -\infty<x<+\infty).\end{cases} \qquad (17)$$

When the problem is formulated in this way, its accurate solution is in the form of Eq. (11) which determines the function $\varphi'(z)$.

At $y = 0$ the values of the function $\bar{z}\varphi''(z)+\psi'(z)$ coincide with the values of an analytical function $z\varphi''(z)+\psi'(z)$ in the region in question, since $z = \bar{z}$ at $y = 0$. From Eq. (2) it is easy to obtain

$$\sigma_{xy}=\mathrm{Im}\,[\bar{z}\varphi''(z)+\psi'(z)]. \qquad (18)$$

From the boundary condition $\sigma_{xy} = 0$ at $y = h$ we have

$$\mathrm{Im}\,[z\varphi''(z)+\psi'(z)]=2h\,\mathrm{Re}\,\varphi''(z) \qquad (19)$$

At $y = 0$ from the condition $\sigma_{xy} = 0$ we have

$$\mathrm{Im}\,[z\varphi''(z)+\varphi'(z)]=0. \qquad (20)$$

Let us denote by $F(\zeta)$ an analytical function corresponding to the function $z\varphi''(z)+\psi'(z)$. After mapping the band $0 \le y \le h$ on the upper semi-plane $\eta > 0$, we obtain the Dirichlet problem with the following boundary conditions:

$$\mathrm{Im}\,F(\xi)=u(\xi) \qquad (-\infty<\xi\leqslant 0)$$
$$\mathrm{Im}\,F(\xi)=0 \qquad (0\leqslant\xi<+\infty). \qquad (21)$$

Using (11) and following [2], we obtain a function $u(\xi)$ in the form

$$u(\xi)=\frac{p}{4}\pi\,[2-A]\,\frac{\xi(\xi+1)}{[\xi^2-\xi A+1]^{3/2}} \qquad \begin{pmatrix}A=a+a^{-1}\\ A\geqslant 2\end{pmatrix}. \qquad (22)$$

The solution of the problem studied will be

$$F(\zeta)=-\frac{p}{4}\,[2-A]\left\{\frac{2\zeta}{\zeta^2-\zeta A+1}+\zeta(\zeta+1)\,\chi(\zeta)\right\}. \qquad (23)$$

It should be pointed out that the function $\chi(\zeta)$ has two branches since it is described by an integral of the following type:

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

The above integral has several branches depending on the sign of a and $\Delta = ac - b^2$. At $-\infty<\xi<(a^{-1})$, $a<\xi<+\infty$, when $\xi^2-\xi(a+a^{-1})+1>0$, we have

$$\chi(\xi)=\frac{1}{(\xi^2-\xi A+1)^{3/2}}\times$$
$$\times\left[2\ln\frac{\sqrt{a\xi-1}+\sqrt{\xi/a-1}}{\sqrt{\xi-a}+\sqrt{\xi-a^{-1}}}-\ln\xi\right]. \qquad (24)$$

At $1/a < \xi < a$, when $\xi^2-\xi(a+1/a)+1<0$, we have

$$\chi(\xi)=\frac{-1}{(\xi^2-\xi A+1)(-\xi^2+\xi A-1)^{1/2}}\times$$
$$\times\left[\arcsin\frac{A-2\xi^{-1}}{\sqrt{A^2-4}}-\arcsin\frac{2\xi-A}{\sqrt{A^2-4}}\right]. \qquad (25)$$

The solution obtained should have no singularities at the crack tips. Assuming $\xi = a - s$ and $\xi = a + s$ and doing some simple transformations, it is easy to prove that this is so.

Figure 5 shows that along a segment of the real axis $1/a \le \xi \le a$ the function $F(\xi)$ has a maximum at the point $\xi = 1$ (which in the plane x, y corresponds to the point $z = 0$).

When l is finite and $h \to \infty$, the following is true:

$$a\approx 1+s+\frac{s^2}{2!}+\frac{s^3}{3!}+\frac{s^4}{4!}$$
$$\frac{1}{a}\approx 1-s+\frac{s^2}{2!}-\frac{s^3}{3!}+\frac{s^4}{4!} \qquad (26)$$

Passing in Eqs. (23) and (24) to the limit at $h \to \infty$ and taking into account Eq. (26), we obtain

$$F(1)\sim(l/h)^2. \qquad (27)$$

Consequently, $F(\zeta) \to 0$ at $h \to \infty$, the solution obtained approaches the solution for a single Griffith crack.

In can be concluded from Fig. 6 that the approximate solution, accurate to 10%, will be valid for ratios $l/h \le 0.5$.

REFERENCES

1. V. M. Kuznetsov, "One case of the equilibrium of a system of cracks in an elasto-brittle material," PMTF [Journal of Applied Mechanics and Technical Physics], no. 5, 1966.

2. N. I. Muskhelishvili, Some Fundamental Problems of the Mathematical Theory of Elasticity [in Russian], 4th Edition, Izd-vo, AN SSSR, 1954.

3. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of Complex Variables [in Russian], 2nd Edition, Fizmatgiz, 1958.

4. G. I. Barenblatt, "Mathematical theory of equilibrium cracks formed in brittle fracture" [in Russian], PMTF, no. 4, 1961.

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